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ON THE MIXING COEFFICIENTS OF PIECEWISE MONOTONIC MAPS

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ABSTRACT

We investigate the mixing coefficients of interval maps satisfying Rychlik's conditions. A mixing Lasota–Yorke map is reverse ϕ-mixing. If its invariant density is uniformly bounded away from 0, it is ϕ-mixing iff all images of all orders are big, in which case it is ψ-mixing. Among β-transformations, non-ϕ-mixing is generic. In this sense, the asymmetry of ϕ-mixing is natural.

0. Introduction

MIXING AND MEASURES OF DEPENDENCE BETWEEN σ-ALGEBRAS. A mixing property of a stationary stochastic process $(\dots, X_{-1}, X_0, X_1, \dots)$ reflects a decay of the statistical dependence between the past σ-algebra $\sigma(\{X_k : k \le 0\})$ and the asymptotic future σ-algebra $\sigma(\{X_k : k \ge n\})$ as $n \to \infty$ and the various mixing properties are described by corresponding measures of dependence between σ-algebras (see [Br]).

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Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ be sub-σ-algebras. We consider the following measures of dependence:

$$\psi(\mathcal{A}, \mathcal{B}) := \sup\{|\frac{|P(A \cap B) - P(A)P(B)|}{P(A)P(B)}| : B \in \mathcal{B}_+, A \in \mathcal{A}_+\},$$

$$\phi(\mathcal{A}, \mathcal{B}) := \sup\{|\frac{|P(A \cap B)}{P(A)} - P(B)| : B \in \mathcal{B}_+, A \in \mathcal{A}_+\},$$

$$\beta(\mathcal{A}, \mathcal{B}) := \sup\{\frac{1}{2} \sum_{a \in \zeta, b \in \xi} |P(a \cap b) - P(a)P(b)| : \zeta \subset \mathcal{A}, \xi \subset \mathcal{B} \text{ finite partitions}\},$$

where, for $\mathcal{C} \subset \mathcal{F}, \mathcal{C}_+ := \{C \in \mathcal{C} : m(C) > 0\}$.

As shown in [Br],

$$\beta(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}) \leq \frac{1}{2}\psi(\mathcal{A}, \mathcal{B}).$$

Note that $\beta(\mathcal{A}, \mathcal{B}) = \beta(\mathcal{B}, \mathcal{A})$ and $\psi(\mathcal{B}, \mathcal{A}) = \psi(\mathcal{A}, \mathcal{B})$ but it may be that $\phi(\mathcal{B}, \mathcal{A}) \neq \phi(\mathcal{A}, \mathcal{B})$ (see [Br] and below).

Accordingly, we set $\phi_+(\mathcal{A}, \mathcal{B}) := \phi(\mathcal{A}, \mathcal{B})$ and $\phi_-(\mathcal{A}, \mathcal{B}) := \phi(\mathcal{B}, \mathcal{A})$.

These measures of dependence give rise to the following **mixing coefficients** of a stationary stochastic process $(\ldots, X_{-1}, X_0, X_1, \ldots)$ defined on the probability space $(\Omega, \mathcal{F}, P)$:

$$\xi(n) := \xi(\sigma(\{X_t\}_{t \leq 0}), \sigma(\{X_t\}_{t \geq n+1}))(\xi = \psi, \phi_+, \phi_-, \beta),$$

and the stationary stochastic process $(\ldots, X_{-1}, X_0, X_1, \ldots)$ is called **ξ-mixing** if $\xi(n) \underset{n \to \infty}{\longrightarrow} 0$.

The property β-mixing is also known as **absolute regularity** and **weak Bernoulli**.

By stationarity, we have

$$\xi(n) = \sup_{k \geq 1} \xi(\sigma(\{X_t\}_{0 \leq t \leq k-1}), \sigma(\{X_t\}_{t \geq n+k+1}))(\xi = \psi, \phi_+, \phi_-, \beta).$$

PIECEWISE MONOTONIC MAPS. A **non-singular, piecewise monotonic map** (PM map) of the interval $X := [0, 1]$ is denoted (X, T, α), where α is a finite or countable collection of open subintervals of X which is a partition in the sense that $\bigcup_{a \in \alpha} a = X \bmod m$ (where m is Lebesgue measure) and $T \colon X \to X$ is a map such that $T|_A$ is absolutely continuous, strictly monotonic for each $A \in \alpha$.

Let (X, T, α) be a PM map. For each $n \geq 1$, (X, T^n, α_n) is also a PM map, where

$$\alpha_n := \{[a_0, \ldots, a_{n-1}] := \bigcap_{k=0}^{n-1} T^{-k} a_k : a_0, \ldots, a_{n-1} \in \alpha\}.$$

A PM map (X,T,α) satisfies $m \circ T^{-1} \ll m$, whence $f \in L^\infty(m) \Rightarrow f \circ T \in L^\infty(m)$. Let $\widehat{T}: L^1(m) \to L^1(m)$ be the predual of $f \mapsto f \circ T$; then

$$\widehat{T}^n g = \sum_{a \in \alpha_n} v_a' 1_{T^n a} g \circ v_a$$

where $v_a: T^n a \to a$ is given by $v_a := (T^n|_a)^{-1}$. Under certain additional assumptions (see below), $\exists h \in L^1(m)$, $h \geq 0$, $\int_X h dm = 1$, so that $\widehat{T}h = h$, i.e. $dP = hdm$ is an absolutely continuous, T-invariant probability (a.c.i.p.).

Mixing coefficients of PM maps. Now let (X,T,α) be a PM map with a.c.i.p. P. We define the **mixing coefficients** with respect to the probability space $(X, \mathcal{B}(X), P)$:

$$\xi(n) := \sup_{k \geq 1} \xi(\sigma(\alpha_0^{k-1}), T^{-(n+k)}\mathcal{B}(X)) \quad (\xi = \psi, \phi_+, \phi_-, \beta)$$

and call the PM map **ξ-mixing** if $\xi(n) \underset{n\to\infty}{\longrightarrow} 0$.

Example: Gauss' continued fraction map. This PM map (X,T,α) with $Tx = 1/x \bmod 1$ and $h(x) = 1/\ln 2(1+x)$ was one of the first considered (see [I-K]). Following Kuzmin's proof (in [Ku]) that $\|\widehat{T}^n 1 - h\|_\infty \leq M\theta^{\sqrt{n}}$ (some $M > 0, \theta \in (0,1)$), Khinchine noted ([Kh]) that

$$\psi^\star(n) := \sup_{a \in \alpha_k, k \geq 1} \|\widehat{T}^n \frac{v_a'}{m(a)} - h\|_\infty \leq M\theta^{\sqrt{n}}.$$

This estimate was improved by Lévy to $\psi^\star(n) \leq M\theta^n$ (see [L]).

To see that (X,T,α) is ψ-mixing, one estimates the similar

$$\psi^\circ(n) := \sup_{a \in \alpha_k, k \geq 1} \frac{1}{m(a)} \|\widehat{T}^n(v_a' h \circ v_a) - hP(a)\|_\infty.$$

The exponential convergence to zero of $\psi^\circ(n)$ was shown in [Go]. Theorem 1(b) is a generalization of this to non-Markov situations. The connection with mixing is seen through the identity $\widehat{T}^{n+k}(1_a h) = \widehat{T}^n(v_a' h \circ v_a)$ for $k, n \geq 1$, $a \in \alpha_k$ (see below).

RU maps. These are PM maps satisfying the conditions (U) and (R) below. The PM map (X,T,α) is called **uniformly expanding** if

(U) $$\inf_{x \in a \in \alpha} |T'(x)| =: \beta > 1;$$

and is said to

- satisfy **Rychlik's condition** ([Ry], see also [ADSZ]) if

(R) $$\sum_{A\in\alpha} \|1_{TA}v_A'\|_{BV} =: \mathcal{R} < \infty,$$

where $\|f\|_{\rm BV} := \|f\|_\infty + \bigvee_X f$ and v_A' is that version of this L_1-function which minimizes variation.

Suppose that (X, T, α) satisfies (R) and (U); then by proposition 1 in [Ry], $\exists M_0 > 0, \theta \in (0,1)$ such that

(1) $$\sum_{a\in\alpha_n} \|1_{T^n a}v_a' f\circ v_a\|_{\rm BV} \le M_0(\theta^n\|f\|_{BV} + \|f\|_1) \le 2M_0\|f\|_{BV} \quad \forall n \ge 1.$$

It follows (see [Ry]) that the ergodic decomposition of $(X, \mathcal{B}(X), m, T)$ is finite, that each ergodic component is open $\bmod m$ and that the tail $\bigcap_{n=0}^\infty T^{-n}\mathcal{B}$ is finite and cyclic on each ergodic component. Moreover, on each ergodic component $C, \exists h = h_C: C \to \mathbb{R}$, $h \ge 0$, $\int_X hdm = 1$, $h \in BV$, $[h > 0]$ open $\bmod m$ so that $\widehat{T}h = h$ and $C = \bigcap_{n=0}^\infty T^{-n}[h > 0] \bmod m$. The probability $dP_C = h_C dm$ is an ergodic a.c.i.p. for T. A RU map (X, T, α) which is conservative and ergodic with respect to m is called **basic**. In this case, $\exists h: X \to \mathbb{R}$, $h > 0$, $\int_X hdm = 1$, $h \in BV$ so that $\widehat{T}h = h$. A RU map (X, T, α) which is conservative and ergodic with respect to m is called **weakly mixing** if $(X, \mathcal{B}(X), P, T)$ is weakly mixing (where $P \sim m$ is the a.c.i.p. for T).

Let (X, T, α) be a weakly mixing RU map; then it is exact with respect to m, and $\exists C > 0$, $r \in (0,1)$ so that

(2) $$\|\widehat{T}^n f - \int_X fdmh\|_{\rm BV} \le Cr^n\|f\|_{\rm BV} \quad \forall f \in {\rm BV},\ n \ge 1$$

([Ry], see also the earlier [H-K] for the case where $\#\alpha < \infty$).

AFU MAPS. The PM map (X, T, α) is called C^2 if, for all $A \in \alpha$, $T: \overline{A} \to T\overline{A}$ is a C^2 diffeomorphism. The C^2 PM (X, T, α) map is called an **AFU map** (as in [Z]) if it satisfies (U),

(F) $$T\alpha := \{TA : A \in \alpha\} \text{ is finite;}$$

and

(A) $$\sup_X \frac{|T''|}{(T')^2} < \infty.$$

A **Lasota–Yorke (LY) map** is an AFU map (X, T, α) with α finite (as in [L-Y]).

Let (X, T, α) be an AFU map; then as can be gleaned from [Z],

- (X, T, α) is a RU map whose ergodic components are finite unions of intervals,
- $1_{[h>0]}\frac{1}{h} \in \mathrm{BV}$ (and $\frac{1}{h} \in \mathrm{BV}$ in case (X, T, α) is basic) whenever $h \in BV$, $h \geq 0, \widehat{T}h = h$,
- $\exists K > 0$ so that

$$(3) \qquad |v_a''(x)| \leq K v_a'(x), \quad v_a'(x) = e^{\pm K} \frac{m(a)}{m(T^n a)} \quad \forall n \geq 1,\ a \in \alpha_n.$$

1. Mixing coefficients of RU maps

As shown in [Ry], if (X, T, α) is a weakly mixing RU map, then

$$\beta(n) \leq 2CM_0 \|h\|_{\mathrm{BV}} r^n \quad (n \geq 1)$$

where M_0 is as in (1) and $C > 0, r \in (0,1)$ are as in (2) (see the remark after Lemma 2 below).

THEOREM 1:

(a) *Let (X, T, α) be a weakly mixing RU map.*
If $\inf_{[h>0]} h > 0$, then $\exists B > 0$ so that $\phi_-(n) \leq Br^n$.

(b) *Let (X, T, α) be a basic, weakly mixing AFU map.*
If $\inf_{n\geq 1, a\in\alpha_n} m(T^n a) > 0$, then $\exists B > 0$ so that $\psi(n) \leq Br^n$.

(c) *Let (X, T, α) be a PM map with a.c.i.p. $P \ll m, \#\alpha < \infty$ and so that $\sigma(\{T^{-n}\alpha : n \geq 0\}) = \mathcal{B}(X)$.*
If $\inf_{n\geq 1, a\in\alpha_n} m(T^n a) = 0$, then $\phi_+(n) = 1 \forall n \geq 1$.

Remarks: (1) It follows that $\phi_-(n) \to 0$ exponentially for any weakly mixing AFU map. This result was announced in [Go] for β-transformations.

(2) Part (b) of the theorem is only established for basic maps as we do not know whether $\inf_{n\geq 1, a\in\alpha_n} m(T^n a) > 0$ implies

$$\inf\{m(T^n a \cap [h > 0]) : n \geq 1, a \in \alpha_n, m(a \cap [h > 0]) > 0\} > 0.$$

PROOF OF THEOREM 1.

LEMMA 2: *Let (X, T, α) be a weakly mixing RU map; then*

$$|P(A \cap T^{-(n+k)}B) - P(B)P(A)| \leq Mr^n m(B \cap [h > 0])$$

$\forall n,k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{B}(X)$, where $M := 2CM_0\|h\|_{\mathrm{BV}}$ with M_0 as in (1) *and $C > 0$, $r \in (0,1)$ as in* (2).

Proof: We show first that

(4)
$$\phi^{\circ}_{-}(n) := \sup\{\|\widehat{T}^{n+k}(1_A h) - P(A)h\|_\infty : k \geq 1, A \in \sigma(\alpha_k)\} \leq Mr^n \quad \forall n \geq 1.$$

The sequence $\phi^{\circ}_{-}(n)$ is the analogue of $\psi^{\circ}(n)$ for ϕ_--mixing.

To see (4), fix $k \geq 1$ and suppose that $A \in \sigma(\alpha_k)$, $A = \bigcup_{a \in \mathfrak{a}} a$ where $\mathfrak{a} \subseteq \alpha_k$; then

$$\widehat{T}^k(h1_A) = \sum_{a \in \mathfrak{a}} v_a' 1_{T^k a} h \circ v_a,$$
$$P(A) = \sum_{a \in \mathfrak{a}} P(a) = \sum_{a \in \mathfrak{a}} \int_{T^k a} v_a' h \circ v_a dm,$$

and for $n \geq 1$,

$$\widehat{T}^{n+k}(1_A h) = \sum_{a \in \mathfrak{a}} \widehat{T}^n(v_a' 1_{T^k a} h \circ v_a).$$

Thus

$$\begin{aligned}\|\widehat{T}^{n+k}(1_A h) - P(A)h\|_\infty &\leq \sum_{a \in \mathfrak{a}} \|\widehat{T}^n(v_a' 1_{T^k a} h \circ v_a) - h\int_{T^k a} v_a' h \circ v_a dm\|_\infty \\ &\leq Cr^n \sum_{a \in \mathfrak{a}} \|v_a' 1_{T^k a} h \circ v_a\|_{\mathrm{BV}} \quad \text{by (2)} \\ &\leq 2CM_0\|h\|_{\mathrm{BV}} r^n \quad \text{by (1)},\end{aligned}$$

establishing (4). Using (4), for $k \geq 1$, $A \in \alpha_k$, $B \in \mathcal{B}$,

$$\begin{aligned}|P(A \cap T^{-(n+k)}B) - P(A)P(B)| &= \left|\int_{B \cap [h>0]} (\widehat{T}^{n+k}(1_A h) - P(A)h) dm\right| \\ &\leq Mr^n m(B \cap [h > 0]). \quad \blacksquare\end{aligned}$$

Remark: It was shown in [Ry], using a version of Lemma 2, that for a weakly mixing RU map, $\beta(n) \leq Mr^n$.

Proof of Theorem 1 (a)*:* By Lemma 2, for $n,k \geq 1$, $A \in \sigma(\alpha_k)$, $B \in \mathcal{B}(X)$,

$$\begin{aligned}|P(A \cap T^{-(n+k)}B) - P(B)P(A)| &\leq Mr^n m(B \cap [h > 0]) \\ &\leq M\left\|\frac{1}{h}\right\|_{L^\infty([h>0])} r^n P(B). \quad \blacksquare\end{aligned}$$

Proof of Theorem 1 (b): We show first that if $\inf_{n\geq 1, a\in\alpha_n} m(T^n a) =: \eta > 0$, then

$$\|\widehat{T}^{n+N}(1_a h) - P(a)h\|_\infty \leq M_1 r^n m(a) \quad \forall N,\ n \geq 1,\ a \in \alpha_N \tag{5}$$

where $M_1 := 4Ce^K\|h\|_{\mathrm{BV}}/\eta$. A standard calculation shows that

$$\|\widehat{T}^k(1_a h)\|_{\mathrm{BV}} = \|v_a' 1_{T^k a} h \circ v_a\|_{\mathrm{BV}} \leq 4e^K\|h\|_{\mathrm{BV}} \frac{m(a)}{m(T^k a)} \quad \forall k \geq 1,\ a \in \alpha_k. \tag{6}$$

To see (5), fix $N, n \geq 1, a \in \alpha_N$, and note that $\widehat{T}^N(1_a h) = 1_{T^N a} h \circ v_a v_a'$. By (2), (3) and (6),

$$\begin{aligned}\|\widehat{T}^{n+N}(1_a h) - P(a)h\|_\infty &= \|\widehat{T}^n(1_{T^N a} h \circ v_a v_a') - P(a)h\|_\infty \\ &\leq Cr^n\|1_{T^N a} h \circ v_a v_a'\|_{\mathrm{BV}} \\ &\leq 4Ce^K\|h\|_{\mathrm{BV}} \frac{m(a)}{m(T^N a)} r^n \leq M_1 r^n m(a).\end{aligned}$$

Now let $N \geq 1$ and suppose that $A \in \sigma(\alpha_N)$, $A = \bigcup_{a\in\mathfrak{a}} a$ where $\mathfrak{a} \subseteq \alpha_N$. It follows from (5) that $\forall B \in \mathcal{B}$, $n \geq 1$,

$$\begin{aligned}|P(A \cap T^{-(n+N)}B) - P(A)P(B)| &\leq \int_B |\widehat{T}^{n+N}(1_A h) - P(A)h|dm \\ &\leq m(B)\|\widehat{T}^{n+N}(1_A h) - P(A)h\|_\infty \\ &\leq M_1 r^n m(B) \sum_{a\in\mathfrak{a}} m(a) \\ &= M_1 r^n m(A) m(B) \\ &\leq M_1 \left(\left\|\frac{1}{h}\right\|_\infty\right)^2 r^n P(A)P(B). \qquad \blacksquare\end{aligned}$$

Proof of Theorem 1 (c): Since $\sigma(\{T^{-n}\alpha : n \geq 0\}) = \mathcal{B}(X)$,

$$\max\{m(a) : a \in \alpha_n\} \to 0 \quad \text{as } n \to \infty.$$

Fix $0 < \epsilon < 1$ and choose $\ell \geq 1$ so that $\max\{m(a) : a \in \alpha_\ell\} < \epsilon/2\|h\|_\infty$.

We show that $\phi_+(N) \geq 1 - \epsilon \forall N \geq 1$. Indeed, fix $N \geq 1$.

Since $\inf_{n\geq 1, a\in\alpha_n} m(T^n a) = 0$, and $\#\alpha_{N+\ell} < \infty$, $\exists k \geq 1$, $\omega \in \alpha_k$ so that $m(T^k\omega) < \min\{m(a) : a \in \alpha_{N+\ell}\}$.

Since $T^k\omega$ is an interval, $\exists b = [b_1, \ldots, b_{N+\ell}]$, $c = [c_1, \ldots, c_{N+\ell}] \in \alpha_{N+\ell}$ so that $T^k\omega \subset J := b \uplus c$. Next, $T^N J \subset [b_{N+1}, \ldots, b_{N+\ell}] \cup [c_{N+1}, \ldots, c_{N+\ell}] \in \sigma(\alpha_\ell)$, whence $m(T^N J) \leq 2\max\{m(a) : a \in \alpha_\ell\} < \epsilon/\|h\|_\infty$ and $\exists B \in \sigma(\alpha_\ell)$, $B \cap T^N J = \emptyset$, $m(B) > 1 - \epsilon/\|h\|_\infty$. It follows that $P(B) > 1 - \epsilon$, and that

$$\phi_+(N) \geq P(B) - P(T^{-(N+k)}B|\omega) = P(B) > 1 - \epsilon. \qquad \blacksquare$$

2. Examples

β-EXPANSIONS. For $\beta > 1$, consider (X, T_β, α) where $X = [0,1]$, T_β: $X \to X$ is given by $T_\beta x := \{\beta x\}$ and

$$\alpha := \left\{\left[\frac{j}{\beta}, \frac{j+1}{\beta}\right)\right\}_{j=0}^{[\beta]-1} \cup \left\{\left[\frac{[\beta]}{\beta}, 1\right)\right\}.$$

As shown in [Pa], (X, T_β, α) is a basic LY map. Theorem 1(a) applies and $\phi_-(n) \to 0$ exponentially. To apply Theorem 1(b), we prove:

PROPOSITION 3:

$$\inf_{n \geq 1, a \in \alpha_n} m(T_\beta^n a) = \inf_{n \geq 1} T_\beta^n 1. \tag{7}$$

Proof: Define (as in [Pa]) π: $X \to \{0, 1, \ldots, [\beta]\}^{\mathbb{N}}$ by $\pi(x)_n := [\beta T_\beta^{n-1} x]$ and set $X_\beta := \overline{\pi_\beta(I)}$. Let $\prec$ denote lexicographic order on $\{0, 1, \ldots, [\beta]\}^{\mathbb{N}}$. Then (see [Pa]) $x < y$ implies $\pi(x) \prec \pi(y)$ and that $\pi(x) \prec \pi(y)$ implies $x \leq y$.

Let

$$\omega = \omega_\beta := \begin{cases} \overline{(a_1, a_2, \ldots, a_{q-1}, a_q - 1)}, & \pi_\beta(1) = (a_1, a_2, \ldots, a_{q-1}, a_q, \overline{0}), \\ \pi_\beta(1) & \text{else.} \end{cases}$$

By [Pa],

$$X_\beta = \{y \in \{0, 1, \ldots, [\beta]\}^{\mathbb{N}} : y_k^\infty \prec \omega \forall k \geq 1\}, \tag{8}$$

where $y_k^\infty := (y_k, y_{k+1}, \ldots)$. For $a = [a_1, \ldots, a_N] \in \alpha_N$, define

$$K_N(a) := \begin{cases} 0 & \nexists 1 \leq n \leq N,\ a_{N-n+1}^N = \omega_1^n \\ \max\{1 \leq n \leq N, a_{N-n+1}^N = \omega_1^n\} & \text{else} \end{cases}$$

(where $a_j^k := (a_j, a_{j+1}, \ldots, a_k)$). Then by (8),

$$\pi([a_1, \ldots, a_N]) = \{x \in X_\beta : x \prec \omega_{K_N(a)+1}^\infty\} = \pi[0, T_\beta^{K_N(a)} 1),$$

whence $T_\beta^N [a_1, \ldots, a_N] = [0, T_\beta^{K_N(a)} 1)$. The proposition follows from this. ∎

Thus by Theorem 1(b), T_β is ψ-mixing iff $\inf_{n \geq 1} T_\beta^n 1 > 0$, or equivalently (see [Bl]), X_β is **specified** in the sense that $\exists L \geq 1$ so that

$$m(a \cap T_\beta^{-(i+L)} b) > 0 \quad \forall a \in \alpha_i,\ b \in \alpha_j.$$

As shown in [S], the set $\{\beta > 1 : X_\beta \text{ specified}\}$ is a meagre set of Lebesgue measure zero and Hausdorff dimension 1 in $\mathbb{R}$ and so exponential ψ-mixing occurs for many $\beta > 1$ for which X_β is not sofic.

"JAPANESE" CONTINUED FRACTIONS. Fix $\alpha \in (0,1]$ and define $T = T_\alpha : [\alpha - 1, \alpha) \circlearrowleft$ by

$$T_\alpha x := \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right| + 1 - \alpha\right].$$

These maps generalize the well-known Gauss map T_1. For $\alpha \in (0,1)$, T_α is a topologically mixing, basic AFU map, whence by Theorem 1,

- exponentially reverse ϕ-mixing, and
- exponentially ψ-mixing when $\inf\{m(T^n a) : n \geq 1, a \in \alpha_n\} > 0$.

Theorem 1(c) does not apply since $\#\alpha = \infty$. However, as shown in [N-N], for Lebesgue a.e. $\frac{1}{2} < \alpha < 1$, T_α is not ϕ_+-mixing.

References

[ADSZ] J. Aaronson, M. Denker, O. Sarig and R. Zweimüller, *Aperiodicity of cocycles and conditional local limit theorems,* Stochastics and Dynamics **4** (2004), 31–62.

[Bl] F. Blanchard, *β-expansions and symbolic dynamics,* Theoretical Computer Science **65** (1989), 131–141.

[Br] R. C. Bradley, *Basic properties of strong mixing conditions,* in *Dependence in Probability and Statistics (Oberwolfach, 1985)*, Progress in Probability and Statistics **11**, Birkhäuser Boston, Boston, MA, 1986, pp. 165–192.

[Go] M. I. Gordin, *Stochastic processes generated by number-theoretic endomorphisms* (English), Soviet Mathematics Doklady **9** (1968), 1234–1237 (translation from Russian: Doklady Akademii Nauk SSSR **182** (1968), 1004–1006).

[H-K] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations,* Mathematische Zeitschrift **180** (1982), 119–140.

[I-K] M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions* (Mathematics and its Applications, **547**), Kluwer Academic Publishers, Dordrecht, 2002.

[Kh] A. Ya. Khintchine, *Continued Fractions,* P. Noordhoff, Ltd., Groningen, 1963 (translation from Russian: *Cepnye drobi*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad, 1935).

[Ku] R. Kuzmin, *On a problem of Gauss,* Doklady Akademii Nauk SSSR, Seriya A (1928), 375–380 (Russian; French translation in: Atti Congresso Internaz. Mat. (Bologna, 1928), Tom VI, Zanichelli, Bologna, 1932, pp. 83–89).

[L-Y] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations,* Transactions of the American Mathematical Society **186** (1973), 481–488.

[L] P. Lévy, *Théorie de l'addition des variables aléatoires* (French), (Monogr. d. probabilit. calcul d. probabilit. et ses appl. 1), XVII, Gauthier-Villars, Paris, 1937.

[N-N] H. Nakada and R. Natsui, *Some strong mixing properties of a sequence of random variables arising from α-continued fractions*, Stochastics Dynamics **3** (2003), 463–476.

[Pa] W. Parry, *On the β-expansions of real numbers*, Acta Mathematica Academiae Scientiarum Hungaricae **11** (1960), 401–416.

[Ry] M. Rychlik, *Bounded variation and invariant measures*, Studia Mathematica **76** (1983), 69–80.

[S] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergodic Theory and Dynamical Systems **17** (1997), 675–694.

[Z] R. Zweimüller, *Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points*, Nonlinearity **11** (1998), 1263–1276.